Does the fact that the Brown-Nash-von Neumann dynamics, and the Smith dynamics, do not always eliminate strictly dominated strategies from the population show that the individual learning rules which yield them are irrational?

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What makes a dynamic that picks a strategy irrational?

A standard answer: agents should evolve out of using strategies that are

> STRICTLY DOMINATED STRATEGIES (SDS)
> A strategy σ is *strictly dominated* if and only if there exists another strategy μ which guarantees the player a higher payoff than σ, regardless of the strategies adopted by other players.
> (Alexander 2001, p.22)

That is, evolutionary dynamics should eliminate SDS because they guarantee a lower payoff if facing the strategy that strictly dominates it. Replicator dynamics do this. However, the innovative dynamics of Brown-Nash-von Neumann (BNN) (Brown & von Neumann 1950) and Smith (Smith 1984) don't: SDS may be preserved among populations (§1). So, because the learning rules preserve SDS, they're irrational (§2). Case closed?

No. Eliminating SDS is desirable for rationality, but not necessary. And here, I'd like to persuade you that rationality is preserved because innovative dynamics compensate by generating benefits unavailable to replicator dynamics (§3). Point being: other nifty features that benefit rationality come at the cost of not eliminating SDS.

1 Brown-Nash-von Neumann and Smith Dynamics

Suppose there's a single unit mass population of agents. I'll consider symmetric two-player games with a finite set of pure strategies $S = \{1, ..., n\}$. Let $x_i(t)$ denote the proportion of agents who select strategy $i \in S$ at time t. The complete state of the system at t is represented by the vector $x(t) = (x_i(t))_{i \in S}$.

A strategy's fitness is defined by a payoff function $F : X \to \Re^n$, where $F_i(x)$ is the payoff of strategy i at population state x_i. Additionally, $\overline{F}(x) = \Sigma_i x_i F_i(x)$ denotes the average population payoff.

A key assumption: agents are boundedly rational concerning payoff information. That is, they'll adopt any above-average strategy–not just the best response. The strategy-revision process is defined by the switch rate ρ_{ij}, which is the rate at which an i-strategist switches to a j-strategist. The rate is conditional on the population state and the payoffs of the game. Following Mertikopoulos & Viossat (2022), I'll bifurcate ρ. The *selection rate* p_{ij} is the probability of an i-strategist considering j. The *adoption rate rij*: the probability of an i-strategist adopting j proportional to a quantity r_{ij} that depends on payoff considerations (p.7).

Generally, there's a two-step process of strategy revision: (1) an agent meets another agent at random and plays a strategy; (2) agents revise their strategy with a probability that depends on the dynamic.

What are the dynamics? Let's look at three. First, Taylor & Jonker (1978)'s *replicator dynamic*. Step (1): after an agent meets another agent at random and plays j with probability x_j. This restricts possible strategy change to strategies that already exist in the population (i.e., if $x_j = 0$ then $\rho_{ij} = 0$). Call this NO INNOVATION. Step (2): they decide whether to imitate with a probability proportional pairwise imitation rule, where $\rho_{ij} = x_j[F_j - F_i]_+$. That is, a rate proportional to the payoff difference. Twofold upshot of (2): (i) every vertex of a game's simplex is a rest point, even if they don't correspond to Nash equilibria; and, (ii) strictly dominated strategies are always eliminated because the multiplicative x_j causes any strategy with below-average fitness to shrink proportionally to its own frequency (Samuelson & Zhang 1992).

Both the BNN and Smith dynamics are *innovative*: strategies that don't already exist in the population may appear. This is because of differences in both steps (1) and (2). The BNN is an innovative dynamic. For step (1), revising i-strategists pick a strategy j at random from a list of possible strategies (giving us innovation). Step (2): the agent adopts it with probability proportional to if j is greater than the population average[1]: $\rho_{ij} = [F_j - \overline{F}]_+$.

And for the Smith dynamic: step (1) is the same as for BNN; for step (2), the agent adopts j with a probability proportional to how much it exceeds the agents own strategy i: $\rho_{ij} = [F_j - F_i]_+$. By considering only the individual (and jettisoning x_i requirement of replicator dynamics), this dynamic retains innovation and doesn't risk being affected by poorly performing strategies because it doesn't consider the population average like BNN.

2 The Problem

As stated, the main issue is that innovative dynamics mightn't push SDS to extinction. Therefore, there must be a problem with the revision protocols. Hofbauer & Sandholm (2011) show that if a dynamic satisfies the four conditions

[1] The agent's current strategy payoff doesn't matter.

of *continuity*, *positive correlation*, *Nash stationarity*, and *Innovation* allow SDS to persist from most initial conditions. Both BSS and Smith dynamics satisfy all four. So, they don't eliminate SDS.

Standard argument: if SDS persists, then the revision protocols are irrational.

3 Is it Irrational to let SDS Persist?

So innovative dynamics let SDS persist. Does this render them irrational? I don't think so. The problem turns on the standard by which we assess rationality. Specifically, I think that total elimination of SDS isn't required . And we can see why this is the case by looking at how the changes made for innovative dynamics at both step (1) and (2) of revision protocols generate Nash Stationarity.

A dynamic satisfies Nash Stationarity iff the population is at rest on a Nash equilibrium of the underlying game. The rationality of this requirement is intuitive: if a population is at a fixed state, then no agent should have incentive to change; if there is incentive, then the dynamic has failed.

To see this, consider the

PRISONER'S DILEMMA

	Cooperate	Defect
Cooperate	R, R	S, T
Defect	T, S	P, P

where $T > R > P > S$. The unique Nash equilibrium is All-Defect.

However, All-Cooperate is a rest point for replicator dynamics ($x_C = 1, x_D = 0$, leaving us with a rate of change of zero because x_D is the multiplicative factor). Because of NO INNOVATION, when $x_D = 0$, there is no defector to imitate. So, the population remains at an exploitable state because of the initial conditions. The replicator could remove the multiplicative, but then they lose payoff-ordering of per-capita growth-rates.

The point I'd like to make here is that the evolutionary dynamic cannot always discover the superior strategy. If we're only dealing with biological matters, then I'll grant this is fine. But evolutionary game theory deals with social matters, and agents need to innovate in social situations, and this is a virtue provided by innovator dynamics. So, do replicator dynamics seem irrational?

4 Conclusion

Not really. However, by proposing this irrationality postulate to Satisfy Nash Stationarity only shows one thing: learning rules for replicator dynamics aren't too good in some scenarios; vice versa for innovators with SDS. But that doesn't make them irrational, because there is a structural trade-off: the very features that allow innovative dynamics to escape non-Nash rest points (innovation, decoupling switch rates from population shares) are precisely what prevents them from eliminating strictly dominated strategies, and these points are jointly incompatible with SDS.

References

Alexander, J. M. (2001), 'Evolutionary game theory', *Standord Encyclopedia of Philosophy* .

Brown, G. W. & von Neumann, J. (1950), Solutions of games by differential equations, *in* H. W. Kuhn & A. W. Tucker, eds, 'Contributions to the Theory of Games', Vol. 24 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, pp. 73–79.

Hofbauer, J. & Sandholm, W. H. (2011), 'Survival of dominated strategies under evolutionary dynamics', *Theoretical Economics* **6**(3), 341–377.
URL: *https://onlinelibrary.wiley.com/doi/abs/10.3982/TE771*

Mertikopoulos, P. & Viossat, Y. (2022), 'Survival of dominated strategies under imitation dynamics', *Journal of Dynamics and Games* **9**(4), 499–528.
URL: *http://dx.doi.org/10.3934/jdg.2022021*

Samuelson, L. & Zhang, J. (1992), 'Evolutionary stability in asymmetric games', *Journal of Economic Theory* **57**(2), 363–391.
URL: *https://www.sciencedirect.com/science/article/pii/002205319290041F*

Smith, M. J. (1984), 'The stability of a dynamic model of traffic assignment: An application of a method of lyapunov', *Transportation Science* **18**, 245–252.

Taylor, P. D. & Jonker, L. B. (1978), 'Evolutionary stable strategies and game dynamics', *Mathematical Biosciences* **40**(1), 145–156.
URL: *https://www.sciencedirect.com/science/article/pii/0025556478900779*